JOURNAL OF COMPUTATIONAL PHYSICS 3, 544–546 (1969)

NOTE

A Direct Numerical Solution to a One-Dimensional Helmholtz Equation

In many physical problems it is necessary to solve the differential equation

$$y_{xx} + f(x)\,y = g(x) \tag{1}$$

in some domain X given two boundary conditions. If $g(x)$ is given in simple functions, a solution is readily obtained. If $g(x)$ is given, however, only as data at discrete points, x_j, $j = 0(1)\,n + 1$, then it may be necessary to apply a numerical technique. There are numerous techniques available. However, in this special case, it is possible to construct the analytical solution to the difference equations when $f(x)$ is a constant, a commonly encountered case.

Let us replace (1) by the difference analog

$$y_{j+1} - 2y_j + y_{j-1} + (\Delta x^2) f y_j = g_j, \tag{2}$$

where $y_0 = a$; $y_{n+1} = b$; $\Delta x = (x_{n+1} - x_0)/(n + 1)$; f is independent of x, and g_j is known. The problem is to find y_j, $j = 1(1)n$, which satisfy the stated boundary conditions. In matrix form (2) can be written

$$AY = G. \tag{3}$$

A is symmetric, tridiagonal and cyclic; its diagonal element, a_{ii}, is $-2 + f(\Delta x)^2 = \beta$, and the off-diagonal elements are unity. If A^{-1} can be found, then the solution to the problem is

$$Y = A^{-1}G. \tag{4}$$

Define det $(A) = |A| = D_n$. An elementary induction proof demonstrates that

$$D_n = D_1(D_{n-1}) - D_{n-2}, \tag{5}$$

where $D_0 = 1$, $D_1 = \beta$. The subscript on D denotes the order of A, or, physically, the number of unknowns, y_j. Let us denote an element of A as a_{ij} and an element of A^{-1} as a'_{ij}.

THEOREM. *Since A^{-1} is symmetric,*

$$a'_{ij} = (-1)^{i+j}\, D_{i-1} D_{n-j}/D_n \qquad \text{if} \quad i \leqslant j \tag{6}$$

and $D_n \neq 0$.

The theorem may be proven by using (5) and taking the special cases $i = j$ and $i \neq j$ and showing, after some manipulation, that, if we define

$$c_{ij} = \sum_{l=1}^{n} a'_{il} a_{lj}, \tag{7}$$

then

$$c_{ii} = 1, \qquad c_{ij} = 0; \quad \text{if} \quad i \neq j. \tag{8}$$

To solve (2) numerically, it is only necessary to construct the solution

$$y_j = \sum_{l=1}^{n} a'_{jl} g_l \tag{9}$$

after defining D_j, $j = 0(1)n$. The values a'_{jl}, Eq. (6), may or may not be stored, as desired. In any event only half of them need be computed since A^{-1} is symmetric. Note, the elements of A need never be stored since they are not needed for the solution. The use of (5), (6), and (9) to construct the solution, y_j, is quite straightforward. To accelerate the computation, it should be observed that (9) may be written

$$\begin{aligned} w_j &= \sum_{l=1}^{j} (-1)^l D_{l-1} g_l, \\ z_j &= \sum_{l=j+1}^{n} (-1)^l D_{n-l} g_l, \\ y_j &= (-1)^j [D_{n-j} w_j + D_{j-1} z_j]/D_n. \end{aligned} \tag{10}$$

The first two sums, w_j and z_j, can be calculated recursively which greatly reduces the operational count for constructing the solution. No attempt has been made to calculate operational counts for this technique but its simplicity and directness commends it over many well-known numerical approaches for solving this particular two-point boundary value problem.

A reviewer kindly disclosed the work by Parker in Lowan [1] which gives the roots of D_n as

$$\beta_k = 2 \cos \frac{k\pi}{n+1}, \qquad k = 1(1)n. \tag{11}$$

If $\beta = \beta_k$, $D_n = 0$, A is singular, and there is no unique solution to (3). However, (11) represents all the roots of D_n and thus, if $|\beta| > 2$, we are confident that A is not singular and a solution may be found to (3). If $|\beta| < 2$, we can always choose Δx such that $\beta \neq \beta_k$; however, for large n, this may still lead to computational difficulties, since the β_k become dense in the small domain $[-2, 2]$.

The above scheme was used in a physical problem concerned with wave motion in a meteorological application. Two equations like (1) from a set of 4 were solved with excellent results hundreds of times during the evolution of a simulated atmospheric disturbance. As a test example for this paper, a numerical example was chosen with $f = \pm\alpha^2$; $g = x^3$; and $y(0) = 1$; $y(1) = y_{n+1} = 0$ for $10^{-2} < \alpha^2 < 10^2$; $1 < n < 10^4$. The results were compared with the analytical solution. For negative α^2, the actual error decreases rapidly with order 10^{-2n} and no difficulty was noticed since $|\beta| > 2$. For positive α^2, (1) becomes pathological as α increases. Even under these conditions, the actual error decreases rapidly with increasing n. Figure 1 shows the approximate distribution of the actual error as a function of α and n. For large α, the true solution oscillates numerous times in the inverval [0, 1] with amplitude 2. For small n, (10) gives large errors; however, as n increases (10) gives excellent agreement with the true solution.

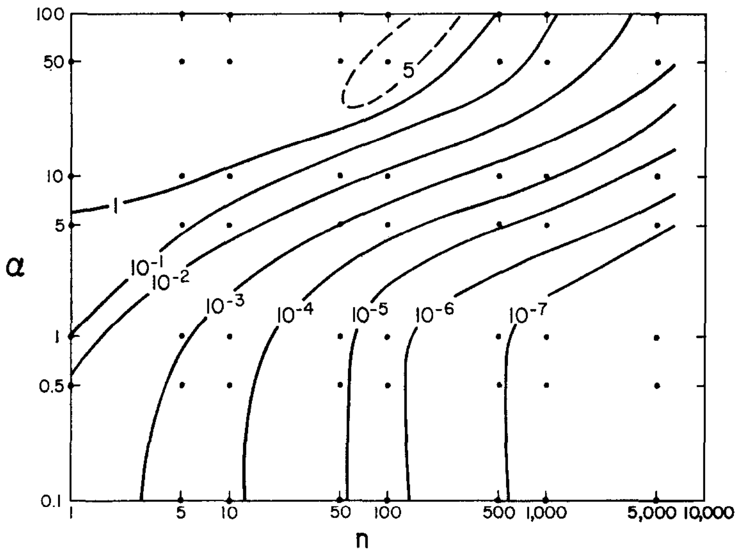

FIG. 1. Approximate error distribution as a function of α and n for $f = +\alpha^2$. True solution ranges from -2 to $+2$. The dots indicate data points for constructing the figure.

REFERENCE

1. A. W. LOWAN, The Operator Approach to Problems of Stability and Convergence of Solutions of Difference Equations and the Convergence of Various Iteration Procedures. *Script. Math. Studia*, **8** (1957).

JAMES J. O'BRIEN

National Center for Atmospheric Research
Boulder, Colorado 80302